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## RULES FOR CONSTRUCTING HYPERPERFECT NUMBERS

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### 1. INTRODUCTION

As usual, let  $\sigma(n)$  denote the sum of all the divisors of  $n$  [with  $\sigma(1) = 1$ ] and let  $\omega(n)$  denote the number of different prime factors of  $n$  [with  $\omega(1) := 0$ ]. The set of prime numbers will be denoted by  $\mathcal{P}$ . The set of hyperperfect numbers (HP's) is the set  $M := \bigcup_{n=1}^{\infty} M_n$ , where

$$M_n := \{m \in \mathbf{N} \mid m = 1 + n[\sigma(m) - m - 1]\}. \quad (1)$$

We also define the sets

$${}_k M_n := \{m \in M_n \mid \omega(m) = k\}, \quad k, n \in \mathbf{N}, \quad (2)$$

and  ${}_k M := \bigcup_{n=1}^{\infty} {}_k M_n$ ; clearly, we have  $M_n = \bigcup_{k=1}^{\infty} {}_k M_n$ . We will also use the related set  $M^* := \bigcup_{n=1}^{\infty} M_n^*$ , where

$$M_n^* := \{m \in \mathbf{N} \mid m = 1 + n[\sigma(m) - m]\}, \quad (3)$$

and the sets

$${}_k M_n^* := \{m \in M_n^* \mid \omega(m) = k\}, \quad k \in \mathbf{N} \cup \{0\}, \quad n \in \mathbf{N}, \quad (4)$$

and  ${}_k M^* := \bigcup_{n=1}^{\infty} {}_k M_n^*$ , so that also  $M_n^* = \bigcup_{k=0}^{\infty} {}_k M_n^*$ .

It is not difficult to verify that  ${}_1 M_n = \emptyset, \forall n \in \mathbf{N}$ , and that

$$\left\{ \begin{array}{l} {}_0 M_n^* = \{1\}, \forall n \in \mathbf{N} \text{ and} \\ {}_1 M_n^* = \begin{cases} \{(n+1)^\alpha, \alpha \in \mathbf{N}\}, & \text{if } n+1 \in \mathcal{P}, \\ \emptyset, & \text{if } n+1 \notin \mathcal{P}. \end{cases} \end{array} \right. \quad (5)$$

$M_1$  is the set of perfect numbers [for which  $\sigma(m) = 2m$ ]. The  $n$ -hyperperfect numbers  $M_n$ , introduced by Minoli and Bear [1], are a meaningful generalization of the even perfect numbers because of the following rule.

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RULE 0 (from [2]): If  $p \in \mathcal{P}$ ,  $\alpha \in \mathbf{N}$ , and if  $q := p^{\alpha+1} - p + 1 \in \mathcal{P}$ , then  $p^\alpha q \in M_{p-1}$ .

There are 71 hyperperfect numbers below  $10^7$  (see [2], [3], [4], [5]). Only one of them belongs to  ${}_3M$ , all others are in  ${}_2M$ . In [6] and [7] the present author has constructively computed several elements of  ${}_3M$  and two of  ${}_4M$ .

In Section 2 of this paper, we shall give rules by which one may find (with enough computer time) an element of  ${}_{(k+2)}M_n$  and of  ${}_{(k+1)}M_n$  from an element of  ${}_kM_n^*$  ( $k \geq 0$ ), and an element of  ${}_kM_n^*$  from an element of  ${}_{(k-2)}M_n^*$  ( $k \geq 2$ ). Because of (5), this suggests the possibility to construct HP's with  $k$  different prime factors for any positive integer  $k \geq 2$ . By actually applying the rules, we have found many elements of  ${}_3M$ , seven elements of  ${}_4M$ , and one element of  ${}_5M$ .<sup>1</sup>

In Section 3, necessary and sufficient conditions are given for numbers of the form  $p^\alpha q$ ,  $\alpha \in \mathbf{N}$ , to be hyperperfect. For example, for  $\alpha \geq 3$ , these conditions imply that there are no other HP's of the form  $p^\alpha q$  than those characterized by Rule 0. The results of this section enable us to compute very cheaply *all* HP's of the form  $p^\alpha q$  below a given bound. Unfortunately, we have not been able to extend these results to more complicated HP's like those of the form  $p^\alpha q^\beta$ ,  $\alpha \geq 2$  and  $\beta \geq 2$ , or  $p^\alpha q^\beta r^\gamma$  with  $\alpha \geq 1$ ,  $\beta \geq 1$  and  $\gamma \geq 1$ , etc. (However, these numbers are extremely scarce compared to HP's of the form  $p^\alpha q$ , and no HP's of the form  $p^\alpha q^\beta$  and  $p^\alpha q^\beta r^\gamma$  with  $\alpha \geq 2$  and  $\beta \geq 2$  have been found to date.)

Because of the importance of the set  $M^*$  for the construction of hyperperfect numbers, we given in Section 4 the results of an exhaustive search for all  $m \in M^*$  with  $m \leq 10^8$  and  $\omega(m) \geq 2$ . It turned out that elements of  ${}_3M^*$  are very rare compared with  ${}_2M^*$ , in analogy with the sets  ${}_3M$  and  ${}_2M$ . This search also gave all elements  $\leq 10^8$  of  $M$ , at very low cost, because of the similarity of the equations defining  $M^*$  and  $M$ . See note 1 below.

The paper concludes with a few remarks, in Section 5, on a possible generalization of hyperperfect numbers to so-called hypercycles, special cases of which are the ordinary perfect numbers and the amicable number pairs.

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<sup>1</sup>*Lists of these numbers may be obtained from the author on request.*

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Remark: After completing this paper, the author computed, with the rules given in Section 2, 860 HP's below the bound  $10^{10}$ . See note 1 above.

### 2. RULES FOR CONSTRUCTING HYPERPERFECT NUMBERS

We have found the following rules [we write  $\bar{a}$  for  $\sigma(a)$ ]:

RULE 1: Let  $k \in \mathbf{N}$ ,  $n \in \mathbf{N}$ ,  $a \in {}_k M_n^*$ , and  $p := n\bar{a} + 1 - n$ ; if  $p \in \mathcal{P}$ , then  $ap \in {}_{(k+1)} M_n$ .

RULE 2: Let  $k \in \mathbf{N} \cup \{0\}$ ,  $n \in \mathbf{N}$ ,  $a \in {}_k M_n^*$ , and  $p := n\bar{a} + A$ ,  $q := n\bar{a} + B$ , where  $AB = 1 - n + n\bar{a} + n^2\bar{a}^2$ ; if  $p \in \mathcal{P}$  and  $q \in \mathcal{P}$ , then  $apq \in {}_{(k+2)} M_n$ .

RULE 3: Let  $k \in \mathbf{N} \cup \{0\}$ ,  $n \in \mathbf{N}$ ,  $a \in {}_k M_n^*$ , and  $p := n\bar{a} + A$ ,  $q := n\bar{a} + B$ , where  $AB = 1 + n\bar{a} + n^2\bar{a}^2$ ; if  $p \in \mathcal{P}$  and  $q \in \mathcal{P}$ , then  $apq \in {}_{(k+2)} M_n^*$ .

The proofs of these rules don't require much more than the application of the definitions, and are therefore left to the reader. In fact, the proof of Rule 2 was already given in [7], although the rule itself was formulated there less explicitly.

Rule 1 can be applied for  $k \geq 1$ , but not for  $k = 0$ , since  ${}_0 M_n^* = \{1\}$  and  $a = 1$  gives  $p = 1 \notin \mathcal{P}$ . For  $k = n = 1$ , Rule 1 reads:

$$\text{If } p := 2^{\alpha+1} - 1 \in \mathcal{P}, \text{ then } 2^\alpha p \in {}_2 M_1,$$

which is Euclid's rule for finding even perfect numbers. For  $k = 1$ , Rule 1 is equivalent to Rule 0, given in Section 1.

Rules 2 and 3 can both be applied for  $k \geq 0$ . For instance, for  $k = 0$ , le 2 reads:

$$\begin{aligned} \text{Let } n \in \mathbf{N} \text{ be given; if } p := n + A \in \mathcal{P} \text{ and } q := n + B \in \mathcal{P}, \\ \text{where } AB = 1 + n^2, \text{ then } pq \in {}_2 M_n. \end{aligned}$$

For  $n = 1, 2$ , and  $6$ , this yields the hyperperfect numbers  $2 \times 3$ ,  $3 \times 7$ , and  $7 \times 43$ , respectively. Rule 3 reads, for  $k = 0$ :

$$\begin{aligned} \text{Let } n \in \mathbf{N} \text{ be given; if } p := n + A \in \mathcal{P} \text{ and } q := n + B \in \mathcal{P}, \\ \text{where } AB = 1 + n + n^2, \text{ then } pq \in {}_2 M_n^*. \end{aligned}$$

For  $n = 4$  and  $n = 10$ , we find that  $7 \times 11 \in {}_2 M_4^*$  and  $13 \times 47 \in {}_2 M_{10}^*$ , respectively.

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Rule 3 shows a rather curious "side-effect" for  $k \geq 1$ : if both the numbers  $p$  and  $q$  in this rule are prime, then not only  $apq \in {}_{(k+2)}M_n^*$ , but also the number  $b := pq$  is an element of  ${}_2M_{n\bar{a}}^*$ . Indeed, we have

$$\begin{aligned} \frac{b-1}{\sigma(b)-b} &= \frac{pq-1}{p+q+1} = \frac{n^2\bar{a}^2 + n\bar{a}(A+B) + AB - 1}{2n\bar{a} + A + B + 1} \\ &= \frac{n^2\bar{a}^2 + n\bar{a}(A+B) + n\bar{a} + n^2\bar{a}^2}{2n\bar{a} + A + B + 1} = n\bar{a} \in \mathbf{N}. \end{aligned}$$

For example, we know that  $7 \times 11 \in {}_2M_4^*$ . From Rule 3 with  $k = 2$ ,  $n = 4$ , and  $a = 7 \times 11$ , we find that  $7 \times 11 \times 547 \times 1291 \in {}_4M_4^*$ ; the side-effect is that

$$547 \times 1291 \in {}_2M_{(4 \times 8 \times 12)}^* = {}_2M_{384}^*.$$

In [6] we gave the following additional rule.

**RULE 4:** Let  $t \in \mathbf{N}$  and  $p := 6t - 1$ ,  $q := 12t + 1$ ; if  $p \in \mathcal{P}$  and  $q \in \mathcal{P}$ , then  $p^2q \in {}_2M_{(4t-1)}$ .

For example,  $t = 1$  and  $t = 3$  give  $5^2 13 \in {}_2M_3$  and  $17^2 37 \in {}_2M_{11}$ , respectively. In Section 3 we will prove that with Rules 1, 2, and 4 it is possible to find all HP's of the form  $p^\alpha q$ ,  $\alpha \in \mathbf{N}$ , below a given bound. We leave it to interested readers to discover why there is no rule (at least for  $k \geq 1$ ), analogous to Rule 1, for finding an element of  ${}_{(k+1)}M_n^*$  from an element of  ${}_kM_n^*$ .

From Rules 1-3, it follows that elements of  ${}_kM_n$  for some given  $k \in \mathbf{N}$  may be found from  ${}_{(k-1)}M_n^*$  (with Rule 1) and from  ${}_{(k-2)}M_n^*$  (with Rule 2) provided that sufficiently many elements of  ${}_{(k-1)}M_n^*$  resp.  ${}_{(k-2)}M_n^*$  are available; these can be found with Rule 3 and the "starting" sets  ${}_0M_n^*$  and  ${}_1M_n^*$  given in (5). We have carried out this "program" for the constructive computation of HP's with three, four, and five different prime factors.

(i) *Construction of elements of  ${}_3M_n$ .* With Rule 1, we found 34 HP's of the form  $pqr$ , from numbers  $pq \in {}_2M_n^*$ :

the smallest is  $61 \times 229 \times 684433 \in {}_3M_{48}$ ;

the largest one is  $9739 \times 13541383 \times 1283583456107389 \in {}_3M_{9732}$ .

The elements of  ${}_2M_n^*$  were "generated" with Rule 3 from  ${}_0M_n^* = \{1\}$ . Using Rule 2 we found, from prime powers  $p^\alpha \in {}_1M_n^*$ , 67 HP's of the form  $pqr$ :

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five of the smallest are given in [6],

the largest is  $8929 \times 79727051 \times 577854714897923 \in {}_3M_{8928}$ ;

48 HP's of the form  $p^2qr$ ,

the smallest five are given in [6],

the largest is  $7459^2 414994003583 \times 34444004601637408163219 \in {}_3M_{7458}$ ;

9 of the form  $p^3qr$ ,

the smallest is given in [6],

the largest is  $811^3 432596915921 \times 89927962885420066391 \in {}_3M_{810}$ ;

4 of the form  $p^4qr$ ,

the smallest is  $7^4 30893 \times 36857 \in {}_3M_6$ ,

the largest is  $223^4 553821371657 \times 130059326113901 \in {}_3M_{222}$ ;

and, furthermore,

$7^6 1340243 \times 2136143 \in {}_3M_6$ ,

$13^7 815787979 \times 11621986347871 \in {}_3M_{12}$ ,

and

$19^8 322687706723 \times 11640844402910006759 \in {}_3M_{18}$ .

(ii) *Construction of elements of  ${}_4M_n$ .* In order to construct elements of  ${}_4M_n$  with Rule 1, sufficiently many elements of  ${}_3M_n^*$  had to be available. This was realized with Rule 3, starting with elements  $p^\alpha \in {}_1M_{(p+1)}$ ,  $p \in \mathcal{P}$ . The following four HP's with four different prime factors were found:

$3049 \times 9297649 \times 69203101249 \times 5981547458963067824996953 \in {}_4M_{3048}$ ,

$4201 \times 17692621 \times 7061044981 \times 2204786370880711054109401 \in {}_4M_{4200}$ ,

$181^2 5991031 \times 579616291 \times 20591020685907725650381 \in {}_4M_{180}$ ,

$181^3 1108889497 \times 33425259193 \times 39781151786825440683346549261 \in {}_4M_{180}$ .

By means of Rules 2 and 3, the following three additional elements of  ${}_4M_n$  were found:

$1327 \times 6793 \times 10020547039 \times 17769709449589 \in {}_4M_{1110}$  (is in [6]),

$1873 \times 24517 \times 79947392729 \times 80855915754575789 \in {}_4M_{1740}$  (is in [7]),

$5791 \times 10357 \times 222816095543 \times 482764219012881017 \in {}_4M_{3714}$ .

(iii) *Construction of an element of  ${}_5M_n$ .* We have also constructively computed one element of  ${}_5M_n$  with Rule 1. The elements of  ${}_4M_n^*$  needed for this purpose were computed from  ${}_0M_n^*$  by twice applying Rule 3 (first yielding elements of  ${}_2M_n^*$ , then elements of  ${}_4M_n^*$ ). The HP found is the largest

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one we know of (apart from the ordinary perfect numbers). It is the 87-digit number:

$$\begin{aligned} &209549717187078140588332885132193432897405407437906414 \\ &236764925538317339020708786590793 \\ &= 4783 \times 83563 \times 1808560287211 \times 297705496733220305347 \\ &\times 973762019320700650093520128480575320050761301 \in {}_5M_{4524}. \end{aligned}$$

### 3. CHARACTERIZATION OF ALL HP'S OF THE FORM $p^\alpha q$

The hyperperfect numbers of the form  $p^\alpha q$  are characterized by the following theorem.

Theorem: Let  $m := p^\alpha q$  ( $\alpha \in \mathbf{N}$ ,  $p \in \mathcal{P}$ ,  $q \in \mathcal{P}$ ) be a hyperperfect number, then

- (i)  $\alpha = 1 \Rightarrow (\exists n \in \mathbf{N}$  with  $m \in {}_2M_n$  such that  $p = n + A$ ,  $q = n + B$ , with  $AB = 1 + n^2$ );
- (ii)  $\alpha = 2 \Rightarrow (\exists t \in \mathbf{N}$  with  $m \in {}_2M_{(4t-1)}$  and  $p = 6t - 1$  and  $q = 12t + 1$ )  
 $\vee (m \in {}_2M_{(p-1)}$  with  $q = p^3 - p + 1$ );
- (iii)  $\alpha > 2 \Rightarrow (m \in {}_2M_{(p-1)}$  with  $q = p^{\alpha+1} - p + 1$ ).

Proof: (i) This case follows immediately from Rule 2 (with  $k = 0$ ).

(ii) If  $p^2 q$  is hyperperfect, then the number  $(p^2 q - 1)/((p + 1)(p + q))$  must be a positive integer. Consider the function

$$f(x, y) := \frac{x^2 y - 1}{(x + 1)(x + y)}, \quad x, y \in \mathbf{N}.$$

To characterize all pairs  $x, y$  for which  $f(x, y) \in \mathbf{N}$ , we can safely take  $x \geq 2$  and  $y \geq 2$ . Let  $x \geq 2$  be fixed, then we have for all  $y \geq 2$ ,

$$f(x, y) < \frac{x^2 y}{(x + 1)(x + y)} < \frac{x^2}{x + 1} = x - 1 + \frac{1}{x + 1}.$$

Hence, the largest integral value which could possibly be assumed by  $f$  is  $x - 1$ , and one easily checks that this value is actually assumed for  $y = x^3 - x + 1$ . So we have found

$$f(x, x^3 - x + 1) = x - 1, \quad x \in \mathbf{N}, \quad x \geq 2. \quad (6)$$

One also easily checks that  $f$  is monotonically increasing in  $y$  ( $x$  fixed), so that

$$2 \leq y \leq x^3 - x + 1. \quad (7)$$

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Now, in order to have  $f \in \mathbf{N}$ , it is necessary that  $x + 1$  divides  $x^2y - 1$ , or, equivalently, that  $x + 1$  divides  $y - 1$ , since

$$\frac{x^2y - 1}{x + 1} = y(x - 1) + \frac{y - 1}{x + 1}.$$

Therefore, we have  $y = k(x + 1) + 1$ , with  $k \in \mathbf{N}$  and  $1 \leq k \leq x(x - 1)$  by (7). Substitution of this into  $f$  yields

$$f(x, y) = \frac{kx^2 + x - 1}{(k + 1)(x + 1)} = x - 1 - \frac{x^2 - x - k}{(k + 1)(x + 1)} =: x - 1 - g(x, k).$$

It follows that  $x + 1$  must divide  $x^2 - x - k$ , or, equivalently, that  $x + 1$  must divide  $k - 2$ . Hence,  $k = j(x + 1) + 2$ , with  $j \in \mathbf{N} \cup \{0\}$  and  $0 \leq j \leq x - 2$ . Substitution of this into  $g$  yields

$$g(x, j(x + 1) + 2) = \frac{x - 2 - j}{j(x + 1) + 3}.$$

This function is decreasing in  $j$ , and for  $j = 0, 1, \dots, x - 2$  it assumes the values:

$$\begin{aligned} g(x, 2) &= (x - 2)/3, \\ g(x, x + 3) &= \frac{x - 3}{x - 4} < 1, \\ &\vdots \\ g(x, x(x - 1)) &= 0. \end{aligned}$$

follows that there is precisely one more possibility [in addition to (5)] for  $f$  to be a positive integer, viz., when  $j = 0$ ,  $k = 2$ ,  $y = 2x + 3$ , and  $x \pmod{3} = 2$ . So we have found

$$f(3t - 1, 6t + 1) = 2t - 1, \quad t \in \mathbf{N}. \tag{8}$$

The statement in the Theorem now easily follows from (6) and (8).

(iii) As in the proof of (ii), we now have to find out for which values of  $x, y \in \mathbf{N}$ ,  $x \geq 2$ , and  $y \geq 2$ , the function  $f(x, y) \in \mathbf{N}$ , where

$$f(x, y) := \frac{x^\alpha y - 1}{(x^{\alpha-1} + \dots + 1)(x + y)}, \quad \alpha > 2.$$

For fixed  $x \geq 2$ , we have

$$f(x, y) < \frac{x^\alpha}{x^{\alpha-1} + \dots + 1} = x - 1 + \frac{1}{x^{\alpha-1} + \dots + 1}.$$

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As in the proof of (ii) we find that  $f(x, y) = x - 1$  for  $y = x^{\alpha+1} - x + 1$  and that  $2 \leq y \leq x^{\alpha+1} - x + 1$ . Furthermore,  $x^{\alpha-1} + \dots + 1$  must divide  $x^\alpha y - 1$ , so that  $y = k(x^{\alpha-1} + \dots + 1) + 1$ , with  $1 \leq k \leq x(x-1)$ . Substitution of this into  $f$  yields a certain function  $g$ , in the same way as in the proof of (ii), but in this case  $g$  can only assume integral values for  $k = x(x-1)$ . This implies the statement in the Theorem, case (iii). Q.E.D.

It is easy to see that the characterizations given in this Theorem are equivalent to Rule 2 ( $k = 0$ ) when  $\alpha = 1$ , to Rule 4 or Rule 1 ( $k = 1$ ) when  $\alpha = 2$ , and to Rule 1 ( $k = 1$ ) when  $\alpha > 2$ .

This Theorem enables us to find very cheaply all HP's of the form  $p^\alpha q$ ,  $\alpha \in \mathbf{N}$ , below a given bound. For example, to find all HP's in  $M_n$  of the form  $pq$  below  $10^8$ , we only have to check whether

$$p := n + A \in \mathcal{P} \quad \text{and} \quad q := n + B \in \mathcal{P}$$

for all possible factorizations of  $AB = 1 + n^2$ , for  $1 \leq n \leq 4999$ . This range of  $n$  follows from the fact that if  $pq \in M_n$  then  $pq > 4n^2$ . The following additional restrictions can be imposed on  $n$ :

- (i)  $n$  should be 1 or even since, if  $n$  is odd and  $n \geq 3$ , then  $n^2 + 1 \equiv 2 \pmod{4}$ , so that one of  $A$  or  $B$  is odd and one of  $p$  or  $q$  is even and  $\geq 4$ .
- (ii) If  $n \geq 3$ , then  $n \equiv 0 \pmod{3}$ , since if  $n \equiv 1$  or  $2 \pmod{3}$ , then  $n^2 + 1 \equiv 2 \pmod{3}$ , so that one of  $A$  or  $B$  is  $\equiv 1 \pmod{3}$  and the other is  $\equiv 2 \pmod{3}$ ; consequently, one of  $p$  or  $q$  is  $\equiv 0 \pmod{3}$  and  $> 3$ .

Hence, the only values of  $n$  to be checked are  $n = 1$ ,  $n = 2$ , and  $n = 6t$ ,  $1 \leq t \leq 833$ . It took about 6 seconds CPU-time on a CDC CYBER 175 computer to check these values of  $n$ , and to generate in this way all HP's of the form  $pq$  below  $10^8$ .

### 4. EXHAUSTIVE COMPUTER SEARCHES

From the rules given in Section 2, it follows that it is of importance to know elements of  $M^*$  when one wants to find elements of  $M$ . Therefore,



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we have carried out an exhaustive computer search for all elements of  $M^*$  below the bound  $10^8$ . Because of (5) the search was restricted to elements with at least two different prime factors. A check was done to determine whether  $(m-1)/(\sigma(m)-m) \in \mathbf{N}$ , for all  $m \leq 10^8$  with  $\omega(m) \geq 2$ . Since the most time-consuming part is the computation of  $\sigma(m)$ , a second check was done to determine whether  $(m-1)/(\sigma(m)-m-1) \in \mathbf{N}$  [in the case where  $(m-1)/(\sigma(m)-m) \notin \mathbf{N}$ ]. If so,  $m$  was an HP; thus, our program also produced, almost for free, all HP's below  $10^8$ . (The search took about 100 hours of "idle" computer time on a CDC CYBER 175.) The results are as follows.

Apart from the ordinary perfect numbers, there are 146 HP's below  $10^8$ . Only two of them have the form  $p^\alpha qr$ :

$$13 \times 269 \times 449 \in {}_3M_{12}^* \quad \text{and} \quad 7^2 383 \times 3203 \in {}_3M_6^*;$$

these were also found in the searches described in Section 2. All others have the form characterized in Section 3, and could have been found with a search based on that characterization (using the fact that if  $p^\alpha q \in {}_2M_n$ , then  $p > n$  and  $q > n$ ). A question that naturally arises is the following: Are there any HP's that *cannot* be constructed with one of Rules 1, 2, or 4?<sup>2</sup>

There are 312 numbers  $m \leq 10^8$  which belong to  $M^*$  and which have  $\omega(m)$

2. Of these, 306 have the form  $pq$  and could have been (and, as a check, actually were) found very cheaply with Rule 3 of Section 2. The others are:

$$\begin{aligned} 7 \times 61 \times 229 &\in {}_3M_6^*, & 113 \times 127 \times 2269 &\in {}_3M_{58}^*, \\ 149 \times 463 \times 659 &\in {}_3M_{96}^*, & 19 \times 373 \times 10357 &\in {}_3M_{18}^*, \\ 151 \times 373 \times 1487 &\in {}_3M_{100}^*, & 7 \times 11 \times 547 \times 1291 &\in {}_4M_4^*; \end{aligned}$$

the second, third, and fifth numbers could not have been found using Rule 3.

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<sup>2</sup>The referee has answered this question in the affirmative by giving the example  $12161963773 = 191 \times 373 \times 170711 \in M_{126}^*$ .

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5. HYPERCYCLES

A possible generalization of hyperperfect numbers can be obtained as follows. Let  $n \in \mathbf{N}$  be given, and define the function  $f_n : \mathbf{N} \setminus \{1\} \Rightarrow \mathbf{N}$  as

$$f_n(m) := 1 + n[\sigma(m) - m - 1], m \in \mathbf{N} \setminus \{1\}. \quad (9)$$

Starting with some  $m_0 \in \mathbf{N} \setminus \{1\}$ , one might investigate the sequence

$$m_0, f_n(m_0), f_n(f_n(m_0)), \dots \quad (10)$$

For  $n = 1$ , this is the well-known aliquot sequence of  $m_0$ , which can have cycles of length 1 (perfect numbers), length 2 (amicable pairs), and others. In order to get some impression of the cyclic behavior for  $n > 1$ , we have computed, for  $2 \leq n \leq 20$ , five terms of all sequences (10) with starting term  $m_0 \leq 10^6$ , and we have registered the cycles with length  $\geq 2$  and  $\leq 5$  in the following table.

TABLE 1  
HYPERCYCLES\*

$n$	$k$	$m_0, m_1, \dots, m_{k-1}$
5	2	19461 = $3 \times 13 \times 499$ , 42691 = $11 \times 3881$
7	3	925 = $5^2 \times 37$ , 1765 = $5 \times 353$ , 2507 = $23 \times 109$
8	2	28145 = $5 \times 13 \times 433$ , 66481 = $19 \times 3499$
	3	238705 = $5 \times 47741$ , 381969 = $3^3 \times 43 \times 47$ , 2350961 = $79 \times 29759$
9	4	94225 = $5^2 \times 3769$ , 181153 = $7^2 \times 3697$ , 237057 = $3 \times 31 \times 2549$ , 714737 = $61 \times 11717$
	2	3452337 = $3^2 \times 7 \times 54799$ , 17974897 = $53 \times 229 \times 1481$
10	2	469 = $7 \times 67$ , 667 = $23 \times 29$
	2	1315 = $5 \times 263$ , 2413 = $19 \times 127$
	2	1477 = $7 \times 211$ , 1963 = $13 \times 151$
	2	2737 = $7 \times 17 \times 23$ , 6463 = $23 \times 281$
10	3	1981 = $7 \times 283$ , 2901 = $3 \times 967$ , 9701 = $89 \times 109$
12	2	697 = $17 \times 41$ , 2041 = $13 \times 157$
	2	3913 = $7 \times 13 \times 43$ , 12169 = $43 \times 283$
	2	54265 = $5 \times 10853$ , 130297 = $29 \times 4493$
14	2	1261 = $13 \times 97$ , 1541 = $23 \times 67$
	3	508453 = $11 \times 17 \times 2719$ , 1106925 = $3 \times 5^2 \times 14759$ , 10126397 = $281 \times 36037$

\*Different numbers  $m_0, m_1, \dots, m_{k-1}$  such that  $m_k = m_0$ , where  $m_{i+1} := f_n(m_i)$ ,  $f_n$  defined in (9).

RULES FOR CONSTRUCTING HYPERPERFECT NUMBERS

TABLE 1 (continued)

$n$	$k$	$m_0, m_1, \dots, m_{k-1}$
19	2	9197 = $17 \times 541$ , 10603 = $23 \times 461$
	4	184491 = $3^3 6833$ , 1688493 = $3 \times 562831$ , 10693847 = $709 \times 15083$ , 300049 = $31 \times 9679$
	2	5151775 = $5^2 251 \times 821$ , 24124073 = $89 \times 271057$

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